ON MAGNETIC CURVES IN THE 3-DIMENSIONAL HEISENBERG GROUP

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Abstract. We consider normal magnetic curves in 3-dimensional Heisenberg group $H_3$. We prove that $\gamma$ is a normal magnetic curve in $H_3$ if and only if it is a geodesic obtained as an integral curve of $e_3$ or a non-Legendre slant circle or a Legendre helix or a slant helix. We obtain the parametric equations of normal slant magnetic curves in 3-dimensional Heisenberg group $H_3$.

1. Introduction

Let $(M, g)$ be a Riemannian manifold and $F$ a closed 2-form. Then $F$ is called a magnetic field (see [1], [2] and [8]) if it is associated by the relation

$$g(\Phi X, Y) = F(X, Y), \quad \forall X, Y \in \chi(M) \quad (1.1)$$

to the Lorentz force $\Phi$ which is defined as a skew symmetric endomorphism field on $M$. Let $\nabla$ be the Levi-Civita connection associated to the metric $g$ and $\gamma : I \to M$ a smooth curve. Then $\gamma$ is called a magnetic curve or a trajectory for the magnetic field $F$ if it is solution of the Lorentz equation

$$\nabla_{\gamma'(t)} \gamma'(t) = \Phi(\gamma'(t)). \quad (1.2)$$

The Lorentz equation generalizes the equation of geodesics. A curve which satisfies the Lorentz equation is called magnetic trajectory. It is well-known that the magnetic curves have constant speed. When the magnetic curve $\gamma$ is arc length parametrized, it is called a normal magnetic curve [9].

In [4], magnetic curves in Sasakian 3-manifolds were considered. In [15], the classification of Killing magnetic curves in $S^2 \times \mathbb{R}$ was given. In [16], the authors prove that a normal magnetic curve on the Sasakian sphere $S^{2n+1}$ lies on a totally geodesic sphere $S^3$. In [9], magnetic curves in a $(2n + 1)$-dimensional Sasakian manifold was studied. In [6], Killing magnetic curves in three-dimensional almost paracontact manifolds were considered. In [14], magnetic curves on flat para-Kähler manifolds were studied. In [18], magnetic curves in 3D semi-Riemannian manifolds was considered. In [13], magnetic trajectories in an almost contact metric manifold $\mathbb{R}^{2N+1}$ were studied. Magnetic curves in cosymplectic manifolds were studied in [10]. Periodic magnetic curves in Berger spheres were considered in [12]. Some closed magnetic curves on a 3-torus were investigated in [17].
Moreover, in [19], Legendre curves in 3-dimensional Heisenberg group were investigated.

Motivated by the above studies, in the present paper, we consider normal magnetic curves in 3-dimensional Heisenberg group $H_3$. We prove that $\gamma$ is a normal magnetic curve in $H_3$ if and only if it is a geodesic obtained as an integral curve of $e_3$ or a non-Legendre slant circle with curvature $\kappa = |q| \sin \alpha$ and of constant contact angle $\alpha = \arccos\left(-\frac{\lambda^2}{2q}\right)$, where $-\frac{\lambda^2}{2q} \in [-1, 1]$ or a Legendre helix with $\kappa = |q|$ and $\tau = \frac{\lambda}{2}$ or a slant helix with $\kappa = |q| \sin \alpha$ and $\tau = \frac{\lambda}{2} + \cos \alpha$. Moreover, we obtain the parametric equations of normal slant magnetic curves in 3-dimensional Heisenberg group $H_3$.

2. Preliminaries

Let $M^{2n+1} = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and $\Omega$ the fundamental 2-form of $M^{2n+1}$ defined by

$$\Omega(X, Y) = g(\varphi X, Y). \quad (2.1)$$

If $\Omega = d\eta$, then $M^{2n+1}$ is called a contact metric manifold [3].

The magnetic field $\Omega$ on $M^{2n+1}$ can be defined by

$$F_q(X, Y) = q\Omega(X, Y),$$

where $X$ and $Y$ are vector fields on $M^{2n+1}$ and $q$ is a real constant. $F_q$ is called the contact magnetic field with strength $q$ [13]. If $q = 0$ then the magnetic curves are geodesics of $M^{2n+1}$. Because of this reason we shall consider $q \neq 0$ (see [4] and [9]).

From (2.1) and (1.1), the Lorentz force $\Phi$ associated to the contact magnetic field $F_q$ can be written as

$$\Phi_q = q\varphi.$$

So the Lorentz equation (1.2) can be written as

$$\nabla_{\gamma'(t)} \gamma'(t) = q\varphi(\gamma'(t)), \quad (2.2)$$

where $\gamma : I \subseteq R \to M^{2n+1}$ is a smooth curve parametrized by arc length (see [9] and [13]).

The Heisenberg group $H_3$ can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{H_3} = dx^2 + dy^2 + \eta \otimes \eta,$$

where $(x, y, z)$ are standard coordinates in $\mathbb{R}^3$ and

$$\eta = dz + \frac{\lambda}{2} (y dx - x dy),$$

where $\lambda$ is a non-zero real number. If $\lambda = 1$, then the Heisenberg group $H_3$ is frequently referred as the model space $Nil_3$ of the Nil geometry in the sense of Thurston [20]. The Heisenberg group is a multiplicative group, and this is essential for the construction of a left-invariant orthonormal basis. The readers would acknowledge to know the expression of the product. Since $\lambda \neq 0$, the 1-form $\eta$ satisfies $d\eta \wedge \eta = -\lambda dx \wedge dy \wedge dz$. Hence $\eta$ is a contact form. In [11], J.
Inoguchi obtained the Levi-Civita connection $\nabla$ of the metric $g$ with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{\lambda y}{2} \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{\lambda x}{2} \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}. \tag{2.3}$$

He obtained

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = \frac{\lambda}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{\lambda}{2} e_2,$$

$$\nabla_{e_2} e_1 = -\frac{\lambda}{2} e_2, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = \frac{\lambda}{2} e_1, \quad \nabla_{e_3} e_1 = -\frac{\lambda}{2} e_2, \quad \nabla_{e_3} e_2 = \frac{\lambda}{2} e_1, \quad \nabla_{e_3} e_3 = 0. \tag{2.4}$$

We also have the Heisenberg brackets

$$[e_1, e_2] = \lambda e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

Let $\varphi$ be the $(1, 1)$-tensor field defined by $\varphi(e_1) = e_2, \varphi(e_2) = -e_1$ and $\varphi(e_3) = 0$. Then using the linearity of $\varphi$ and $g$ we have

$$\eta(e_3) = 1, \quad \varphi^2(X) = -X + \eta(X)e_3, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

We also have

$$d\eta(X, Y) = \frac{\lambda}{2} g(X, \varphi Y)$$

for any $X, Y \in \chi(M)$. Then for $\xi = e_3, (\varphi, \xi, \eta, g)$ defines an almost contact metric structure on $H_3$. If $\lambda = 2$, then $(\varphi, \xi, \eta, g)$ is a contact metric structure and the Heisenberg group $H_3$ is a Sasakian space form of constant holomorphic sectional curvature $-3$ (see [11]). For arbitrary $\lambda \neq 0$, we do not work in contact Riemannian geometry. However, the fundamental 2-form is closed and hence it defines a magnetic field.

Let $\gamma: I \to H_3$ be a Frenet curve parametrized by arc length $s$. The contact angle $\alpha(s)$ is a function defined by $\cos\alpha(s) = g(T(s), \xi)$. The curve $\gamma$ is said to be slant if its contact angle $\alpha(s)$ is a constant [7]. Slant curves of contact angle $\frac{\pi}{2}$ are traditionally called Legendre curves [3].

For $(H_3, \varphi, \xi, \eta, g)$, the Lorentz equation (1.2) can be written as

$$\nabla_{\gamma'}(t)\gamma'(t) = q\varphi(\gamma'(t)), \tag{2.5}$$

(see [9]).

### 3. Magnetic Curves in 3-dimensional Heisenberg Group $H_3$

Let $\gamma: I \to H_3$ be a curve parametrized by arc length. We say that $\gamma$ is a Frenet curve if one of the following three cases holds:

i) $\gamma$ is of osculating order 1. In this case, $\nabla_{\gamma'}\gamma' = 0$, which means that $\gamma$ is a geodesic.

ii) $\gamma$ is of osculating order 2. In this case, there exist two orthonormal vector fields $T = \gamma', \ N$ and a positive function $\kappa$ (curvature) along $\gamma$ such that $\nabla_T T = \kappa N, \nabla_T N = -\kappa T$.

iii) $\gamma$ is of osculating order 3. In this case, there exist three orthonormal vector fields $T = \gamma', \ N, B$ and a positive function $\kappa$ (curvature) and $\tau$ (torsion) along $\gamma$ such that

$$\nabla_T T = \kappa N,$$

$$\nabla_T N = -\kappa T + \tau B,$$
\( \nabla_T B = -\tau N, \)

where \( \kappa = \|\nabla_T T\| \). A circle is a Frenet curve of osculating order 2 such that \( \kappa \) is a non-zero positive constant; a helix is a Frenet curve of osculating order 3 such that \( \kappa \) and \( \tau \) are non-zero constants (see [19]).

**Theorem 3.1.** Let \((H_3, \varphi, \xi, \eta, g)\) be the Heisenberg group and consider the contact magnetic field \( F_q \) for \( q \neq 0 \) on \( H_3 \). Then \( \gamma \) is a normal magnetic curve associated to \( F_q \) in \( H_3 \) if and only if

i) \( \gamma \) is a geodesic obtained as an integral curve of \( e_3 \) or

ii) \( \gamma \) is a non-Legendre circle with curvature \( \kappa = |q| \sin \alpha \) and of constant contact angle \( \alpha = \arccos \left( \frac{\lambda}{2q} \right) \), where \( -\frac{\lambda}{2q} \in [-1,1] \) or

iii) \( \gamma \) is a Legendre helix with \( \kappa = |q| \) and \( \tau = \frac{\lambda}{2} \) or

iv) \( \gamma \) is a slant helix with \( \kappa = |q| \sin \alpha \) and \( \tau = \frac{\lambda}{2} + q \cos \alpha \), where \( \alpha \) is a constant such that \( \alpha \in (0, \pi) \).

**Proof.** If the magnetic curve \( \gamma \) is a geodesic, then \( \varphi T = 0 \), which means that \( T \) is collinear to \( e_3 \). Then being unitary we must have \( T = \mp e_3 \). So \( \gamma \) is a geodesic obtained as an integral curve of \( \xi \).

Since \( \gamma \) is parametrized by arc-length, we can write

\[
T = \sin \alpha \cos \beta e_1 + \sin \alpha \sin \beta e_2 + \cos \alpha e_3,
\]

where \( \alpha = \alpha(s) \) and \( \beta = \beta(s) \). Using (2.4) we have

\[
\nabla_T T = \left( \alpha' \cos \alpha \cos \beta - \sin \alpha \sin \beta \left( \beta' - \lambda \cos \alpha \right) \right) e_1
\]
\[
+ \left( \alpha' \cos \alpha \sin \beta + \sin \alpha \cos \beta \left( \beta' - \lambda \cos \alpha \right) \right) e_2
\]
\[
- \alpha' \sin \alpha e_3.
\]

(3.2)

On the other hand, by the use of (3.1), it follows that

\[
\varphi T = -\sin \alpha \sin \beta e_1 + \sin \alpha \cos \beta e_2.
\]

(3.3)

Since \( \gamma \) is a magnetic curve

\[
\nabla_T T = q\varphi(T),
\]

which gives us

\[
\alpha' \cos \alpha \cos \beta - \sin \alpha \sin \beta \left( \beta' - \lambda \cos \alpha \right) = -q \sin \alpha \sin \beta,
\]

(3.4)

\[
\alpha' \cos \alpha \sin \beta + \sin \alpha \cos \beta \left( \beta' - \lambda \cos \alpha \right) = q \sin \alpha \cos \beta,
\]

(3.5)

\[
\alpha' \sin \alpha = 0.
\]

(3.6)

From (3.6), we find \( \alpha' = 0 \) or \( \sin \alpha = 0 \). If \( \sin \alpha = 0 \), then \( \varphi T = 0 \). So by the discussion of the beginning of the proof, it follows that \( \gamma \) is a geodesic obtained as an integral curve of \( e_3 \). If \( \alpha' = 0 \), then \( \alpha \) is a constant, this means that \( \gamma \) is a slant curve. So we can assume that \( \sin \alpha > 0 \), which means that \( \alpha \in (0, \pi) \).

Since \( \alpha \) is a constant, from (3.4) or (3.5), we obtain \( \beta' - \lambda \cos \alpha = q \). Hence

\[
\beta(s) = (\lambda \cos \alpha + q) s + c,
\]

(3.7)

where \( c \) is an arbitrary real number.

Substituting \( \alpha' = 0 \) and \( \beta' - \lambda \cos \alpha = q \) into (3.2), we find

\[
\nabla_T T = -q \sin \alpha \sin \beta e_1 + q \sin \alpha \cos \beta e_2.
\]

(3.8)
Now let \( \{T, N, B\} \) denote the Frenet frame of \( \gamma \). Since \( \nabla_T T = \kappa N \), from (3.8) we obtain
\[
\kappa = |q| \sin \alpha = \text{constant.} \tag{3.9}
\]
By (3.8) and (3.9), it follows that
\[
N = \text{sgn}(q) \left( -\sin \beta e_1 + \cos \beta e_2 \right). \tag{3.10}
\]
Then by the use of (3.10), (2.4) and \( \beta' - \lambda \cos \alpha = q \), we find
\[
\nabla_T N = \text{sgn}(q) \left( -\cos \beta \left( \frac{\lambda}{2} \cos \alpha + q \right) e_1 \right.
\]
\[
\left. -\sin \beta \left( \frac{\lambda}{2} \cos \alpha + q \right) e_2 + \frac{\lambda}{2} \sin \alpha e_3 \right). \tag{3.11}
\]
Now we define the cross product \( \times \) by \( e_1 \times e_2 = e_3 \) and we compute \( B = T \times N \). Then we obtain
\[
B = \text{sgn}(q) \left( -\cos \alpha \cos \beta e_1 - \cos \alpha \sin \beta e_2 + \sin \alpha e_3 \right). \tag{3.12}
\]
If \( \gamma \) is Legendre then from (3.12), it is a Legendre helix with \( \kappa = |q| \) and \( \tau = \frac{\lambda}{2} \). If \( \gamma \) is non-Legendre then from (3.12), it is a slant helix with \( \kappa = |q| \sin \alpha \) and \( \tau = \frac{\lambda}{2} + q \cos \alpha \).

If the osculating order is 2, then from (3.12), \( \cos \alpha = -\frac{\lambda}{2q} \). So \( \gamma \) is a circle with \( \kappa = |q| \sin \alpha \) and of constant contact angle \( \alpha = \arccos(-\frac{\lambda}{2q}) \), where \( -\frac{\lambda}{2q} \in [-1, 1] \).

Conversely, assume that \( \gamma \) is a slant helix with \( \kappa = |q| \sin \alpha \) and \( \tau = \frac{\lambda}{2} + q \cos \alpha \), where \( \alpha \) is the contact angle between \( \gamma \) and \( e_3 \). Then \( \cos \alpha = g(T, e_3) \). Hence \( T \) is of the form (3.1). Taking the covariant derivative of (3.1) with respect to \( T \), since \( \alpha \) is a constant, we have
\[
\nabla_T T = \left( \beta' - \lambda \cos \alpha \right) \left[ -\sin \alpha \sin \beta e_1 + \sin \alpha \cos \beta e_2 \right] = \kappa N
\]
So we find \( g(e_3, N) = 0 \). Hence \( e_3 \) can be written as
\[
e_3 = \cos \alpha T + \mu B, \tag{3.13}
\]
where \( \mu = \mp \sin \alpha \) is a real constant since \( \|e_3\| = 1 \). By (3.13), by a covariant differentiation, we have
\[
\frac{\lambda}{2} \varphi T = (\tau \mu - \kappa \cos \alpha) N, \tag{3.14}
\]
which gives us
\[
\frac{\lambda^2}{4} g(\varphi T, \varphi T) = \frac{\lambda^2}{4} \sin^2 \alpha = (\tau \mu - \kappa \cos \alpha)^2. \tag{3.15}
\]
Since \( \kappa = |q| \sin \alpha \) and \( \tau = \frac{\lambda}{2} + q \cos \alpha \), from the equation (3.15), we find \( \mu = \text{sgn}(q) \sin \alpha \). Then the equality (3.14) turns into
\[
\varphi T = \text{sgn}(q) \sin \alpha N.
\]
Using Frenet formulas
\[
\nabla_T T = \kappa N = |q| \sin \alpha N = q \varphi T.
\]
Then the Lorentz equation (2.5) is satisfied. Hence \( \gamma \) is a magnetic curve.

If \( \gamma \) is a Legendre helix with \( \kappa = |q| \) and \( \tau = \frac{\lambda}{2q} \), then taking \( \alpha = \frac{\pi}{2} \) in the above case, we have
\[
\varphi T = sgn(q)N
\]
and
\[
\nabla_T^2 = \kappa N = |q| N = q\varphi T,
\]
which means that \( \gamma \) is a magnetic curve.

If \( \gamma \) is a non-Legendre circle with curvature \( \kappa = |q| \sin \alpha \) and of constant contact angle \( \alpha = \arccos(-\frac{\lambda}{2q}) \), then taking \( \tau = 0 \) and \( \cos \alpha = -\frac{\lambda}{2q} \) we have again \( \nabla_T^2 = q\varphi T \). This implies that \( \gamma \) is a magnetic curve.

Then we get the result as required. \( \square \)

4. Explicit Formulas for Magnetic Curves in 3-dimensional Heisenberg Group \( H_3 \)

In [5], R. Caddeo, C. Oniciuc and P. Piu obtained the parametric equations of all non-geodesic biharmonic curves in Heisenberg group \( \text{Nil}_3 \). Using the similar method of [5], we can state a result analogous to [Theorem 3.5, [9]]:

**Theorem 4.1.** The normal slant magnetic curves on \( H_3 \), described by (2.2) have the parametric equations

\( a) \)
\[
\begin{align*}
x(s) &= \frac{1}{v} \sin \alpha \sin (vs + c) + d_1, \\
y(s) &= -\frac{1}{v} \sin \alpha \cos (vs + c) + d_2, \\
z(s) &= \left( \cos \alpha + \frac{\lambda}{2v} \sin^2 \alpha \right)s - \frac{\lambda}{2v}d_1 \sin \alpha \cos (vs + c) \\
&\quad - \frac{\lambda}{2v}d_2 \sin \alpha \sin (vs + c) + d_3,
\end{align*}
\]
where \( v = \lambda \cos \alpha + q \neq 0 \) and \( c, d_1, d_2, d_3 \) are real numbers and \( \alpha \) denotes the contact angle which is a constant such that \( \alpha \in (0, \pi) \) or

\( b) \)
\[
\begin{align*}
x(s) &= (\sin \alpha \cos c) s + d_4, \\
y(s) &= (\sin \alpha \sin c) s + d_5,
\end{align*}
\]
and
\[
z(s) = \left( -\frac{q}{\lambda} + \frac{\lambda}{2} \sin \alpha (d_4 \sin c - d_5 \cos c) \right)s + d_6,
\]
where \( c, d_4, d_5 \) and \( d_6 \) are real numbers and \( \alpha \) denotes the contact angle which is a constant such that \( \alpha = \arccos(-\frac{q}{\lambda}) \), where \( -\frac{q}{\lambda} \in [-1, 1] \).

**Proof.** Let \( \gamma(s) = (x(s), y(s), z(s)) \). Then using the equations (2.3), the equation (3.1) can be written as
\[
T = \sin \alpha \cos \beta(s) \left( \frac{\partial}{\partial x} - \frac{\lambda y}{2} \frac{\partial}{\partial z} \right) + \sin \alpha \sin \beta(s) \left( \frac{\partial}{\partial y} + \frac{\lambda x}{2} \frac{\partial}{\partial z} \right) + \cos \alpha \frac{\partial}{\partial z}
\]
(\sin \alpha \cos \beta(s)) \frac{\partial}{\partial x} + (\sin \alpha \sin \beta(s)) \frac{\partial}{\partial y} + \left( \frac{\lambda}{2} x(s) \sin \alpha \sin \beta(s) - \frac{\lambda}{2} y(s) \sin \alpha \cos \beta(s) + \cos \alpha \right) \frac{\partial}{\partial z}, \quad (4.1)

where \beta(s) = (\lambda \cos \alpha + q) s + c. To find the explicit equations, we should integrate the system \( \frac{d\gamma}{ds} = T \). Then using (4.1), we have

\[
\frac{dx}{ds} = \sin \alpha \cos (vs + c),
\]

\[
\frac{dy}{ds} = \sin \alpha \sin (vs + c)
\]

and

\[
\frac{dz}{ds} = \left( \cos \alpha + \frac{\lambda}{2} x(s) \sin \alpha \sin(vs + c) - \frac{\lambda}{2} y(s) \sin \alpha \cos(vs + c) \right),
\]

where \( v = \lambda \cos \alpha + q \).

Assume that \( v \neq 0 \). So the integration of the equations (4.2) and (4.3) gives us

\[
x(s) = \frac{1}{v} \sin \alpha \sin (vs + c) + d_1
\]

and

\[
y(s) = -\frac{1}{v} \sin \alpha \cos (vs + c) + d_2,
\]

where \( d_1 \) and \( d_2 \) are real constants. Then substituting the equations (4.5) and (4.6) in (4.4) we get

\[
\frac{dz}{ds} = \cos \alpha + \frac{\lambda}{2v} \sin^2 \alpha + \frac{\lambda}{2} d_1 \sin \alpha \sin(vs + c) - \frac{\lambda}{2} d_2 \sin \alpha \cos(vs + c).
\]

Hence the solution of the last differential equation gives us

\[
z(s) = \left( \cos \alpha + \frac{\lambda}{2v} \sin^2 \alpha \right) s - \frac{\lambda}{2v} d_1 \sin \alpha \cos(vs + c)
\]

\[
-\frac{\lambda}{2v} d_2 \sin \alpha \sin(vs + c) + d_3,
\]

where \( d_3 \) is a real constant.

Now assume that \( v = \lambda \cos \alpha + q = 0 \). Then \( \alpha = \arccos(-\frac{q}{\lambda}) \), where \(-\frac{q}{\lambda} \in [-1, 1]\). So from (4.2), (4.3) and (4.4), we have

\[
\frac{dx}{ds} = \sin \alpha \cos c,
\]

\[
\frac{dy}{ds} = \sin \alpha \sin c
\]

and

\[
\frac{dz}{ds} = \left( -\frac{q}{\lambda} + \frac{\lambda}{2} x(s) \sin \alpha \sin c - \frac{\lambda}{2} y(s) \sin \alpha \cos c \right). \quad (4.9)
\]

Similar to the solution of the previous case, we find

\[
x(s) = (\sin \alpha \cos c) s + d_4,
\]

\[
y(s) = (\sin \alpha \sin c) s + d_5
\]
and
\[ z(s) = \left( -\frac{q}{\lambda} + \frac{\lambda}{2} \sin \alpha \left( d_4 \sin c - d_5 \cos c \right) \right) s + d_6, \]
where \( d_4, d_5 \) and \( d_6 \) are real constants. This completes the proof of the theorem.

\[ \square \]

Remark 4.1. For \( \lambda = 1 \), the Heisenberg group \( H_3 \) is frequently referred as the model space \( \text{Nil}_3 \). Hence Theorem 3.1 and Theorem 4.1 can be restated taking \( \lambda = 1 \) for the Nil space \( \text{Nil}_3 \).

Acknowledgements

The author would like to thank the referees for their valuable comments, which helped to improve the manuscript.

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Received: March 28, 2017; Revised: August 26, 2017; Accepted: September 15, 2017